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# The complexions of gauge fields

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**Abstract.** The complexion of the electromagnetic field arising in the context of duality rotations is extended to general gauge fields. There are two complexions generally. For non-Abelian gauge fields, a new gauge-invariant Lorentz scalar property described by a non-Abelian complexion is obtained.

## 1. Introduction

Many years ago, Misner and Wheeler discussed a significant Lorentz-invariant scalar property of the electromagnetic field [1]. This property is described by an angle, called the complexion, which measures the ratio of the Lagrangian density and the pseudocharge of the electromagnetic field. Although a given electromagnetic field $F_{\mu\nu}$ uniquely determines the energy-momentum tensor density $T_{\mu\nu}$, one can reconstruct the EM field from a given energy-momentum tensor $T_{\mu\nu}$ only up to a duality rotation (for the meaning of this term, see § 3 below). The additional information needed is the complexion, which fixes the duality rotation and hence the exact EM field $F_{\mu\nu}$. This enables a complete solution of the theory to be obtained for the case in which the electromagnetic field is coupled to gravity.

In this paper we explore a similar property for the general gauge field, and discuss its meaning for a theory in which the general (non-Abelian) gauge field is coupled to gravity. In § 2 we discuss the general properties of $4\times4$ antisymmetric matrices and give a short proof of a property of the EM field. This property plays a crucial role in the 'already unified theory'. Then we turn to general gauge fields in § 3 and find that besides the Abelian complexion similar to that obtained by Misner and Wheeler, there is a new non-Abelian complexion which is particular to non-Abelian gauge fields. One can use it to characterise whether the gauge field is Abelian or not. The final section is a short discussion. Our conclusion is that the theory of a non-Abelian gauge field coupled to gravity is unlikely to be soluble.

## 2. General properties of antisymmetric matrices.

Let us firstly discuss some general properties of any $4\times4$ antisymmetric matrix. These properties shall be used later. The field strength $F^a_{\mu\nu}$ in a gauge field theory is a set of antisymmetric tensors. Here we shall treat them in matrix form and define a matrix $F^a$ with elements

$$(F^a)_{\mu\nu} = F^a_{\mu\nu}. \tag{1}$$

$F^a$ is a $4\times 4$ antisymmetric matrix. From this matrix one can define a Euclidean dual matrix $\tilde{F}^a$ and a Minkowskian dual matrix $^*F^a$, with elements, respectively,

$$(\tilde{F}^a)_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^a_{\alpha\beta}$$
$$(^*F^a)_{\mu\nu} = \tfrac{1}{2}\mathrm{i}\varepsilon_{\mu\nu\alpha\beta}F^a_{\alpha\beta} = \mathrm{i}\tilde{F}^a_{\mu\nu}. \qquad (2)$$

Obviously we have the properties

$$\tilde{\tilde{F}}^a = F^a \qquad *(^*F^a) = -F^a. \qquad (3)$$

Since we are going to discuss duality, it is convenient to expand the matrix $F^a$ over a set of base matrices which are 'eigenmatrices of duality'. The well defined Dirac $\gamma$ matrices are also $4\times 4$ matrices, and among them six matrices can be made antisymmetric. Verifying the duality property, we find they can be divided into two groups

$$\xi_1^1 = \gamma_1 = \gamma_5\gamma_3 C \qquad \xi_2^1 = -\gamma_3 = \gamma_5\gamma_1 C \qquad \xi_3^1 = \mathrm{i}\gamma_1\gamma_3 = i\gamma_5 C$$
$$\xi_1^{\bar{1}} = \mathrm{i}C = \mathrm{i}\gamma_2\gamma_4 \qquad \xi_2^{\bar{1}} = \mathrm{i}\gamma_5\gamma_4 = \mathrm{i}\gamma_5\gamma_2 C \qquad \xi_3^{\bar{1}} = \mathrm{i}\gamma_5\gamma_2 = -\mathrm{i}\gamma_5\gamma_4 C. \qquad (4)$$

They are all purely imaginary, antisymmetric, Hermitian, self-inverse (their square is the unit matrix) and traceless

$$\xi_i^{r+} = \xi_i^r \quad \xi_i^{r2} = 1 \quad \mathrm{Tr}\,\xi_i^r = 0 \qquad i = 1, 2, 3 \quad r = 1, \bar{1}. \qquad (5)$$

The remarkable thing is that they are really 'eigenmatrices' of duality:

$$\tilde{\xi}_i^r = r\xi_i^r \qquad r = 1, \bar{1} \qquad \bar{1} \equiv -1. \qquad (6)$$

Besides, these matrices possess very nice commutation and anticommutation relations

$$[\xi_i^r, \xi_j^s] = 2\mathrm{i}\delta^{rs}\varepsilon_{ijk}\xi_k^r$$
$$\{\xi_i^r, \xi_j^s\} = 2\delta^{rs}\delta_{ij} + 2\delta^{r\bar{s}}\Gamma_{ij}^{(r,\bar{r})}. \qquad (7)$$

Here there is no summation over $r$ on the right-hand side of the equations. $\Gamma_{ij}^{(r,\bar{r})}$ are symmetric matrices, traceless and self-inverse,

$$(\Gamma_{i,j}^{(r,\bar{r})})^{\mathrm{t}} = \Gamma_{i,j}^{(r,\bar{r})} \qquad \Gamma_{i,j}^{(r,\bar{r})} = \Gamma_{j,i}^{(\bar{r},r)} \qquad \mathrm{Tr}\,\Gamma_{i,j}^{(r,\bar{r})} = 0 \qquad (\Gamma_{i,j}^{(r,\bar{r})})^2 = 1. \qquad (8)$$

The notation $\Gamma^{\mathrm{t}}$ means the transpose of $\Gamma$.

From (8) we get the important property of $\xi_i^r$

$$\mathrm{Tr}\,\xi_i^r\xi_j^s = 4\delta^{rs}\delta_{ij}. \qquad (9)$$

Expanding $F^a$ over $\xi_i^r$, we have

$$F^a = F_i^{a,r}\xi_i^r$$
$$F_i^{a,r} = \tfrac{1}{4}\mathrm{Tr}\,(F^a\xi_i^r) \qquad (10)$$

For the dual field $^*F^a$, the expansion is

$$^*F^a = \mathrm{i}rF_i^{a,r}\xi_i^r. \qquad (11)$$

We see from this equation that the duality operation is a sort of reflection.

In Minkowski space we choose the elements to be

$$E_j^a = -\mathrm{i}F_{j4}^a \qquad B_i^a = \tfrac{1}{2}\varepsilon_{ijk}F_{jk}^a \qquad (12)$$

then the expansion coefficients are

$$F_i^{a,r} = \tfrac{1}{2}(E_i^a + \mathrm{i}rB_i^a) \qquad (13)$$

which is a familiar combination.

The $\xi_i^r$ matrices turn out to be what 't Hooft called $\eta$ symbols [2] which are very useful in constructing the instanton solutions and can be introduced in another way [3].

Using (7) and (10), one can easily prove the following identities valid for any $4\times 4$ antisymmetric matrices:

$$F^aF^b - {}^*F^{b*}F^a = \tfrac{1}{2}\,\mathrm{Tr}\,(F^aF^b) = -\tfrac{1}{2}\,\mathrm{Tr}\,({}^*F^{a*}F^b) \tag{14}$$

$$F^{a*}F^b + F^{b*}F^a = \tfrac{1}{2}\,\mathrm{Tr}\,({}^*F^aF^b) = \tfrac{1}{2}\,\mathrm{Tr}\,(F^{a*}F^b) \tag{15}$$

$$\mathrm{Tr}\,(F^2) = -\mathrm{Tr}\,({}^*F)^2 \tag{16}$$

$$F^*F = \tfrac{1}{4}\,\mathrm{Tr}\,(F^*F) \tag{17}$$

$$[F^a, F^b] = -[{}^*F^a, {}^*F^b] \tag{18}$$

$$[F, {}^*F] = 0. \tag{19}$$

To show the identities, we have, for example,

$$\begin{aligned} F^aF^b - {}^*F^{b*}F^a &= F_i^{a,r}F_j^{b,s}(\xi_i^r\xi_j^s + rs\xi_j^s\xi_i^r) \\ &= F_i^{a,r}F_j^{b,s}\tfrac{1}{2}((1-rs)\lfloor \xi_i^r, \xi_j^s] + (1+rs)\{\xi_i^r, \xi_j^s\}) \\ &= F_i^{a,r}F_j^{b,s}(\delta^{r\bar{s}}[\xi_i^r, \xi_j^s] + \delta^{rs}\{\xi_i^r, \xi_j^s\}) \\ &= 2F_i^{a,r}F_j^{b,s}\delta^{rs}\delta_{ij} = 2\,\mathrm{Tr}\,(F^aF^b). \end{aligned}$$

Similar relations can be written down for $F^a$ and $\tilde{F}^a$, only keeping equation (2) in mind. For instance, equation (14) can be transformed into

$$F^aF^b + \tilde{F}^b\tilde{F}^a = \tfrac{1}{2}\,\mathrm{Tr}\,(F^aF^b) = \tfrac{1}{2}\,\mathrm{Tr}\,(\tilde{F}^a\tilde{F}^b). \tag{20}$$

These identities are very useful as long as we discuss the field strength as a whole. For example, a very important property of the energy-momentum tensor of the electromagnetic field [1] can be simply proved, as follows. The energy-momentum tensor of the EM field is

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu\alpha} - \tfrac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta} \tag{21}$$

or, in our matrix notation,

$$T = -F^2 + \tfrac{1}{4}\,\mathrm{Tr}\,F^2 = -({}^*F)^2 - \tfrac{1}{4}\,\mathrm{Tr}\,F^2. \tag{22}$$

The square of $T$ is then

$$\begin{aligned} T^2 &= (-F^2 + \tfrac{1}{4}\,\mathrm{Tr}\,F^2)(-{}^*F^2 - \tfrac{1}{4}\,\mathrm{Tr}\,F^2) \\ &= FF^*F^*F + \tfrac{1}{4}\,\mathrm{Tr}\,(F^2)(F^2 - {}^*F^2) - \tfrac{1}{16}(\mathrm{Tr}\,F^2)^2 \\ &= \tfrac{1}{16}[(\mathrm{Tr}\,F^2)^2 + (\mathrm{Tr}\,(F^*F))^2]. \end{aligned} \tag{23}$$

This equation means that the inverse of $T$ is proportional to itself — a property playing a very crucial role in solving the so-called 'already unified theory' [1]. Here we have proved it by using the identities (14) and (17) in equations (22) and (23).

## 3. The complexions of gauge fields

Now we turn to the duality rotation. A duality rotation of gauge fields $F^a_{\mu\nu}$ is a one-parameter global transformation

$$F^{a'} = F^a\cos\alpha + {}^*F^a\sin\alpha. \tag{24}$$

In terms of the components $F_j^{a,r}$, the above transformation can be written in the following way:

$$F_j^{a,r'} = F_j^{a,r}\cos\alpha + \mathrm{i}rF_j^{a,r}\sin\alpha$$
$$= \exp(\mathrm{i}r\alpha)F_j^{a,r} \qquad \text{(no sum over r)} \tag{25}$$

which is a pure phase transformation, but a different duality index $r$ corresponds to the opposite phase angle.

Since $*(*F^a) = -F^a$, from (24) we obtain the transformed $*F^{a'}$

$$*F^{a'} = -F^a\sin\alpha + *F^a\cos\alpha. \tag{26}$$

If we define a two-dimensional matrix vector

$$f^a \equiv \begin{pmatrix} F^a \\ *F^a \end{pmatrix} \tag{27}$$

then we have the transformation law for $f^a$

$$f^{a'} = \exp(\mathrm{i}\alpha\sigma_2)f^a. \tag{28}$$

This is a two-dimensional rotation, generated by the Pauli matrix $\sigma_2$.

To construct a covariant bilinear quantity, one can use Pauli matrices and unit matrix $\sigma_4 = 1$. Four independent bilinear quantities are

$$I_\mu^{ab} \equiv (f^a)^{\mathrm{t}}\sigma_\mu f^b. \tag{29}$$

Since $\sigma_4$ and $\sigma_2$ commute with $\sigma_2$, the $I_4^{ab}$ and $I_2^{ab}$ are duality invariants

$$I_4^{ab'} = I_4^{ab} \qquad I_2^{ab'} = I_2^{ab}. \tag{30}$$

The quantities $I_1^{ab}$ and $I_3^{ab}$ together form a two-dimensional vector rotating with a double angle ('$2\alpha$ vector'):

$$\begin{pmatrix} I_1^{ab} \\ I_3^{ab} \end{pmatrix} = \exp(\mathrm{i}2\alpha\sigma_2)\begin{pmatrix} I_1^{ab} \\ I_3^{ab} \end{pmatrix}. \tag{31}$$

In the same spirit we can construct covariant trilinear quantities from $f^c$ and $I_\mu^{ab}$. In this case we find that there is no invariant, but have three '$\alpha$ vectors' and one '$3\alpha$ vector'.

The $\alpha$ vectors are

$$J^{abc} = I_4^{ab}f^c \tag{32}$$

$$K^{abc} = I_2^{ab}f^c \tag{33}$$

$$H^{abc} = \begin{pmatrix} H_1^{abc} \\ H_2^{abc} \end{pmatrix}$$

$$H_1^{abc} = \begin{pmatrix} I_1^{ab} \\ I_3^{ab} \end{pmatrix}^t \sigma_1 f^c \qquad H_2^{abc} = \begin{pmatrix} I_1^{ab} \\ I_3^{ab} \end{pmatrix}\sigma_3 f^c \tag{34}$$

$$H^{abc'} = \exp(\mathrm{i}\alpha\sigma_2)H^{abc}. \tag{35}$$

The $3\alpha$ vector is

$$G^{abc} = \begin{pmatrix} G_1^{abc} \\ G_2^{abc} \end{pmatrix}$$

$$G_1^{abc} = \begin{pmatrix} I_1^{ab} \\ I_3^{ab} \end{pmatrix}^t f^c \qquad G_2^{abc} = \begin{pmatrix} I_1^{ab} \\ I_3^{ab} \end{pmatrix}^t \mathrm{i}\sigma_2 f^c \tag{36}$$

$$G^{abc'} = \exp(\mathrm{i}3\alpha\sigma_2)G^{abc}. \tag{37}$$

We may continue to form higher-order duality rotation covariant quantities, for example quadrilinear forms and so on, but it is not necessary because we are interested in determining gauge-invariant and Lorentz-invariant properties, and for this purpose the trilinear combination is high enough.

To form a gauge-invariant quantity, we can employ the group unit tensor $\delta^{ab}$ and the structure constant $f_{abc}$, multiply them by the duality rotation covariants and sum over the group indices (taking the trace in the group space). If we want to form Lorentz invariant quantities, we just take the trace in Minkowski space. Then we find that there are only two duality-covariant, gauge-invariant and Lorentz-invariant quantities; one is bilinear, the other is trilinear.

The bilinear quantity is

$$\begin{aligned} I_1 &= -\tfrac{1}{8}\,\mathrm{Tr}\, I_1^{aa} = \tfrac{1}{4}\,\mathrm{Tr}\,(F^{a*}F^a) = \boldsymbol{E}^a \cdot \boldsymbol{B}^a \\ I_3 &= -\tfrac{1}{8}\,\mathrm{Tr}\, I_3^{aa} = \tfrac{1}{4}\,\mathrm{Tr}\,(F^aF^a) = \tfrac{1}{2}(\boldsymbol{E}^a \cdot \boldsymbol{E}^a - \boldsymbol{B}^a \cdot \boldsymbol{B}^a) \end{aligned} \tag{38}$$

$$I_1' = I_1 \cos 2\alpha - I_3 \sin 2\alpha \qquad I_3' = I_1 \sin 2\alpha + I_3 \cos 2\alpha. \tag{39}$$

From this we can define the first complexion of a gauge field:

$$\begin{aligned} &I_1' = 0 \\ &\alpha_1 = \tfrac{1}{2}\cot^{-1}(I_3/I_1) = \tfrac{1}{2}\cot^{-1}\left(\frac{\boldsymbol{E}^a \cdot \boldsymbol{E}^a - \boldsymbol{B}^a \cdot \boldsymbol{B}^a}{2\boldsymbol{E}^a \cdot \boldsymbol{B}^a}\right). \end{aligned} \tag{40}$$

The meaning of this angle $\alpha_1$ is that, after a duality rotation with such an angle, the gauge fields become 'perpendicular each to other'. We may call $\alpha_1$ the 'Abelian complexion' of gauge fields since it also exists in the U(1) case.

The non-trivial trilinear gauge-invariant, Lorentz-invariant, duality-covariant quantity is, after using the formal identities of $4\times4$ antisymmetric matrices,

$$\begin{aligned} G_4 &= \tfrac{1}{4} f_{abc}\,\mathrm{Tr}\begin{pmatrix} I_1^{ab} \\ I_3^{ab} \end{pmatrix}^{\mathrm{t}} f^c = f_{abc}\,\mathrm{Tr}\,(F^aF^{b*}F^c) \\ &= f_{abc}\varepsilon_{ijk}(3B_i^aB_j^b - E_i^aE_j^b)E_k^c \\ G_2 &= \tfrac{1}{4} f_{abc}\,\mathrm{Tr}\begin{pmatrix} I_1^{ab} \\ I_3^{ab} \end{pmatrix}^{\mathrm{t}} \mathrm{i}\sigma_2 f^c = -f_{abc}\,\mathrm{Tr}\,(F^aF^bF^c) \\ &= f_{abc}\varepsilon_{ijk}(3E_i^aE_j^b - B_i^aB_j^b)B_k^c \end{aligned} \tag{41}$$

$$\begin{aligned} G_4' &= G_4 \cos 3\alpha + G_2 \sin 3\alpha \\ G_2' &= -G_4 \sin 3\alpha + G_2 \cos 3\alpha. \end{aligned} \tag{42}$$

Now we can get the second complexion $\alpha_2$ which can be specified by the vanishing of $G_2'$:

$$\begin{aligned} \alpha_2 &= \tfrac{1}{3}\tan^{-1}(G_2/G_4) \\ &= \tfrac{1}{3}\tan^{-1}\left(\frac{f_{abc}\varepsilon_{ijk}(3E_i^aE_j^b - B_i^aB_j^b)B_k^c}{f_{a'b'c'}\varepsilon_{i'j'k'}(3B_{i'}^{a'}B_{j'}^{b'} - E_{i'}^{a'}E_{j'}^{b'})E_{k'}^{c'}}\right). \end{aligned} \tag{43}$$

Obviously, when the gauge group is Abelian, the quantities $G_2$ and $G_4$ no longer exist, so there is no such complexion. We call this angle $\alpha_2$ the 'non-Abelian complexion', which then sufficiently characterises the non-Abelian property of the gauge theory.

For a Yang-Mills field with symmetry group SU(2), $f_{abc}=\varepsilon_{abc}$ and the non-Abelian complexion $\alpha_2$ is

$$\alpha_2=\tfrac{1}{3}\tan^{-1}\left(\frac{\varepsilon_{abc}\varepsilon_{ijk}(3E_i^aE_j^b-B_i^aB_j^b)B_k^c}{\varepsilon_{lmn}\varepsilon_{pqr}(3B_p^lB_q^m-E_p^lE_q^m)E_r^n}\right). \tag{44}$$

As an example, we consider the Polyakov-'t Hooft monopole solution [4], for which the electric field $E_i^a$ vanishes, but $\varepsilon_{abc}\varepsilon_{ijk}B_i^aB_j^bB_k^c$ is not zero except on the boundary. The non-Abelian complexion angle is then

$$\alpha_2=-\pi/6. \tag{45}$$

## 4. Discussion

In Rainich's already unified theory [1, 5], the electromagnetic field couples to gravity. Due to the crucial property (23) of the stress-energy-momentum tensor, the whole theory can be solved. From the Riemannian curvature tensor of the spacetime, one can construct the energy-momentum tensor, then take a 'Maxwell root' from the obtained stress tensor to get the electromagnetic field strength in the extremal form. After a duality rotation through the complexion angle, the exact electromagnetic field emerges. If a non-Abelian gauge field, instead of the electromagnetic field, is coupled to the gravity, it is unlikely that this whole theory could be solved in terms of a similar technique, since the energy-momentum tensor of a non-Abelian gauge field does not possess a property similar to (23). The fact that the non-Abelian gauge fields have at most two non-vanishing complexions seems to suggest that the additional information is not enough in order to solve the problem completely, since a non-Abelian gauge theory contains at least three different curvatures.

## References

[1] Misner C W and Wheeler J A 1957 *Ann. Phys., NY* **2** 525
[2] 't Hooft G 1976 *Phys. Rev.* D **14** 3432
[3] Prasad M K 1980 *Physica* **1D** 167
[4] Goddard P and Olive D I 1978 *Rep. Prog. Phys.* **41** 1357
[5] Rainich G Y 1925 *Trans. Am. Math. Soc.* **27** 106